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The Effect of Artificial Diffusivity on the Flute Instability

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Abstract

Sometimes, in order to improve the performance of magneto-hydrodynamical codes, artificial diffusivity (D) is introduced in the mass continuity equation. In this communication, an analysis of the effect of the artificial diffusivity on the low-beta plasma stability in a simple geometry is presented. It is shown that, at low diffusivity, one recovers classical results, whereas at high diffusivity the plasma becomes more unstable. Dependence of the stability on D is suppressed if the volume of flux-tube varies insignificantly in the course of the perturbation growth. These observations may help the code runners to identify regimes where the artificial diffusivity is not affecting the results (or vise versa).

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When performing magneto-hydrodynamic (MHD) numerical simulations of the plasma dynamics, one sometimes adds artificial diffusivity to the mass continuity equation (see, e.g., [1]). This allows one to eliminate formation of too steep density gradients and, most importantly, of voids, whose appearance might make a code crash. On the other hand, the presence of this "smoothing" term may affect the results of the simulations and introduce some unphysical effects. In order to address these concerns, we present here an analytic solution of one of the most generic MHD problems, that of the flute stability [2], with the continuity equation containing an artificial diffusivity. The solution presented provides some insights on the possible impact of artificial diffusivity and allows one to identify the situations where this impact may be substantial. We are not aiming at explaining any particular set of simulations for any particular device, but rather provide a general discussion that may serve as a guidance for considering specific cases.

The set of MHD equations used in this communication are as follows:

$$\rho\left[\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right]=-\nabla p+\frac{1}{c}\mathbf{j}\times\mathbf{B}, \tag{1a}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=D\nabla^2\rho, \tag{1b}$$

$$\frac{3}{2}\rho\frac{\partial T}{\partial t}+\frac{3}{2}\rho\mathbf{v}\cdot\nabla T+\rho T\nabla\cdot\mathbf{v}=0, \tag{1c}$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times\mathbf{v}\times\boldsymbol{B}, \tag{1d}$$

$$\nabla\times\boldsymbol{B}=\frac{4\pi}{c}\boldsymbol{j}. \tag{1e}$$

$$p=2\rho T/m_i, \tag{1f}$$

where ρ is the mass density, $\mathbf{v}$ is the plasma velocity field, T is the temperature (common for the electrons and ions), $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{j}$ is the current density, p is the pressure, D is the artificial diffusivity coefficient, and m_i is the ion mass. We are using the CGS (Gaussian) system of units.

Aside from the presence of the artificial diffusivity D in the continuity equation, these are equations of ideal MHD. We have dropped all the dissipative processes (except for the artificial diffusivity) in order to more clearly isolate effects of the finite D. Later on in this communication we will also discuss the role of viscous effects, because an artificial viscosity is also sometimes used in the simulations.

The concerns about the effect of the artificial diffusivity are related to the fact that it is not easy to provide a physics picture that would lead to the appearance of the diffusion term in the continuity equation. One can of course take a viewpoint that it is just a specific model of a particle source, a model for which the source is proportional to $\nabla^2\rho$. However, aside from the somewhat artificial structure of this source, one would then also have to explain why the presence of this source in the continuity equation is not accompanied by the appearance of some sources in the momentum equation, Eq. (1a), and temperature equation, Eq. (1c). Making the sources and sinks of the momentum and particle energy such that the corresponding terms would not show up in either of the aforementioned equations is possible, but it is difficult to associate them with any real sources.

This does *not* mean that the use of the artificial diffusivity would necessarily lead to substantial qualitative errors in the description of the plasma behavior. One can expect that in problems where the plasma compressibility is unimportant, the effect of an even very large D will be weak. Our results support this viewpoint.

Given the lack of a clear physical meaning of the artificial diffusivity, we have taken the following approach. We just apply the set (1) to the analysis of a well-known plasma physics problem, that of the flute instability in a simple geometry, and see what happens. If the solution found does not deviate strongly from the “standard” solution, this signifies that the diffusion term is harmless.

The geometry is illustrated by Fig. 1a. A cylindrically-symmetric plasma, uniform in the axial direction, is immersed in a purely azimuthal magnetic field of the current rod. We consider a low-beta plasma, so that

$$\beta \equiv 8\pi p / B^2 << 1 \tag{2}$$

In the equilibrium, we have a resting plasma, with a uniform density (therefore satisfying the unperturbed continuity equation with artificial diffusivity) and the temperature $T_0(r)$ varying in the radial direction. The unperturbed pressure $p_0(r) = 2\rho_0 T_0(r)/m_i$ (Fig. 2b) varies in the radial direction and serves as a drive for the flute instability [2]. Under such circumstances, one can find a complete analytical solution of the stability problem.

We linearize the set of equations (1) and seek perturbations with the exp(Γt) dependence on time. Denoting the perturbations by the prefix “δ”, and introducing the displacement vector

$$\boldsymbol{\xi} \equiv \delta \mathbf{v} / \Gamma, \tag{3}$$

one obtains from the set (1):

$$\Gamma^2 \rho_0 \boldsymbol{\xi} = -\nabla \delta p + \frac{1}{c} \delta \mathbf{j}_\perp \times \mathbf{B_0}, \tag{4a}$$

$$\delta\rho + \rho_0 \nabla \cdot \boldsymbol{\xi} = \frac{D}{\Gamma} \nabla^2 \delta\rho , \tag{4b}$$

$$\frac{3}{2}\delta T + \frac{3}{2}\boldsymbol{\xi} \cdot \nabla T_0 + T_0 \nabla \cdot \boldsymbol{\xi} = 0 , \tag{4c}$$

$$0 = \nabla \times \boldsymbol{\xi} \times \boldsymbol{B}_0 , \tag{4d}$$

$$\delta p = p_0 \left(\frac{\delta\rho}{\rho_0} + \frac{\delta T}{T_0} \right) \tag{4e}$$

We have neglected the magnetic field perturbation, because, in the pressure-driven modes in a β<<1 plasma, it is of order of β compared to the retained terms.

Flute perturbations in the geometry of Fig. 1a are axisymmetric perturbations with the azimuthal component of the displacement vector $\boldsymbol{\xi}$ equal to zero. Equation (4e) has then a solution:

$$\boldsymbol{\xi} \times \boldsymbol{B}_0 = \nabla \psi , \tag{5}$$

where ψ is a function independent on the azimuth. From Eq. (5) one gets:

$$\boldsymbol{\xi} = \frac{\boldsymbol{B}_0 \times \nabla \psi}{B_0^2} , \; \xi_r = \frac{1}{B_0} \frac{\partial \psi}{\partial z} . \tag{6}$$

By taking the divergence of the first of Eqs. (6), one easily finds that

$$\nabla \cdot \boldsymbol{\xi} = - \frac{2 \xi_r B_0'}{B_0} , \tag{7}$$

where the prime denotes the differentiation over the radius. Substituting this result into Eq. (4c), one finds that

$$\delta T = -\xi_r \left(T_0' - \frac{4 B_0'}{3 B_0} T_0 \right) , \tag{8}$$

whereas Eq. (4b) yields:

$$\delta\rho - \frac{D}{\Gamma} \nabla^2 \delta\rho = \rho_0 \frac{2 \xi_r B_0'}{B_0} . \tag{9}$$

We now consider the first of the equations of the set (4), which yields:

$$\delta \boldsymbol{j}_\perp = \frac{c}{B_0^2}\boldsymbol{B_0}\times(\nabla\delta p + \Gamma^2\rho_0\boldsymbol{\xi}) = \frac{c}{B_0^2}[(\boldsymbol{B_0}\times\nabla\delta p) - \Gamma^2\rho_0\nabla\psi]. \tag{10}$$

Using the charge neutrality condition $\nabla\cdot\delta\mathbf{j}\equiv\nabla\cdot\delta\mathbf{j}_\perp = 0$, one finds from this equation that

$$-\Gamma^2\rho_0\left(\frac{\partial^2\psi}{\partial z^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{2}{r}\frac{\partial\psi}{\partial r}\right) = 2\frac{\partial B_0}{\partial r}\frac{\partial\delta p}{\partial z} \quad . \tag{11}$$

At this point, we make an assumption that, in the perturbations,

$$\partial/\partial z >> \partial/\partial r >> | p_0'/p_0 |. \tag{12}$$

It is known from the theory of the flute instability that this condition corresponds to the fastest growing modes [2, 3]. Under condition (12), Eq. (11) is reduced to

$$-\Gamma^2\rho_0\frac{\partial^2\psi}{\partial z^2} = 2\frac{\partial B_0}{\partial r}\frac{\partial\delta p}{\partial z}. \tag{13}$$

or, according to the second of Eqs. (6), to

$$-\Gamma^2\rho_0\xi_r = \frac{2}{B_0}\frac{\partial B_0}{\partial r}\delta p \quad . \tag{14}$$

For the z dependence of the $\exp(ik_z z)$ type, Eq. (9) yields:

$$\frac{\delta\rho}{\rho_0} = \frac{2\xi_r B_0'}{B_0}\frac{\Gamma}{\Gamma+\nu}, \tag{15}$$

where

$$\nu \equiv k_z^2 D . \tag{16}$$

This quantity is an inverse diffusion time over the spatial scale of perturbations, $D \equiv 1/k_z$.

Then, from Eq. (4e) and Eqs. (8) and (15), one gets:

$$\frac{\delta p}{p_0} = \xi_r\left[-\frac{p_0'}{p_0} + \frac{2B_0'}{B_0}\left(\frac{2}{3} + \frac{\Gamma}{\Gamma+\nu}\right)\right] \quad . \tag{17}$$

Finally, from Eqs. (14) and (17) one obtains the dispersion relation:

$$\Gamma^2 = -\frac{2B_0'}{B_0}\frac{p_0}{\rho_0}\left[-\frac{p_0'}{p_0} + \frac{2B_0'}{B_0}\left(\frac{2}{3} + \frac{\Gamma}{\Gamma+\nu}\right)\right] \tag{18}$$

In the case of a zero diffusion coefficient (or, equivalently, the case of a zero ν), the dispersion relation acquires a standard form for the flute instability of a gas with γ=5/3. The stability criterion can in this case be presented as

$$p_0' U' + \frac{\gamma p_0}{U} U'^2 > 0, \tag{19}$$

where U is specific volume of the flux-tube, $U = \oint dl / B_0 = (\pi c / I) r^2$, and I is the current through the rod (Fig. 1a).

In the opposite limit of a very large diffusion coefficient (i.e., at $\nu \to \infty$), one again recovers the stability criterion (19), although with a different γ, γ=2/3. In other words, the stability criterion becomes more restrictive. Of course, γ cannot be treated in the latter case as a normal adiabatic index, which must be greater than one; in our case, it is just a parameter of the solution.

As we have already mentioned, dispersion relation (18) is a local dispersion relation: it describes the evolution of perturbations with the length-scale smaller than the gradient scale length $| p_0 / p_0' |$.To get more quantitative information about the growth rate, it is convenient to present Eq.(18) in a dimensionless form. Assume that we are studying the instability in the vicinity of the surface $r=a$.We introduce the dimensionless growth-rate, $\hat{\Gamma}$, the dimensionless pressure gradient, $g \equiv -ap_0' / p_0$, and the dimensionless ν:

$$\hat{\Gamma} = \Gamma a / 2s; \;\; g = -3ap_0' / 10 p_0; \;\; \hat{\nu} = \nu a / 2s. \tag{20}$$

where $s \equiv \sqrt{5 p_0 / 3\rho_0}$ is the adiabatic sound speed. Numerical coefficients in (20) are chosen in such a way as to make g=1 correspond to the critical gradient in the “ideal” case (D=0). Note that this instability is very fast: for $\hat{\Gamma}$=1, the dimensional e-folding time is $a/2s$.

The dimensionless dispersion relation reads as:

$$\hat{\Gamma}^2 = g - 1 + \frac{0.6\hat{\nu}}{\hat{\Gamma} + \hat{\nu}} \qquad (21)$$

One can show that, at g>0.4, this equation has one unstable root with a purely exponential growth. The other two roots are stable and, depending on the parameters, are either exponentially damping, or have an oscillatory component. At g<0.4 all roots are stable; some of them exhibit an oscillatory damping.

The unstable domain can be split in two sub-domains, 0.4< g<1, and g >1, with the growth-rate behaving quite differently (Fig. 2). In the first sub-domain, the growth rate is zero at ν=0 (i.e., at zero diffusivity). At small but finite values of ν, the growth rate is proportional to ν and can be approximately represented as

$$\hat{\Gamma} = \hat{\nu}\frac{g - 0.4}{1 - g}. \qquad (22)$$

(this expression is valid for g not too close to the "standard" instability boundary, g=1). The growth rate is proportional to the diffusivity – a feature of what would normally be called "negative energy mode." In our case, as we have g<1 (i.e., we are below the threshold for the "standard" flute instability), the potential energy perturbation is actually positive. The occurrence of the analog of a "negative energy mode" is related here to the aforementioned fact that the artificial diffusivity cannot be easily interpreted in terms of its effect on the energy conservation.

Note that the instability at $0.4<g<1$ is present even at very small values of the diffusivity. Asymptotically, at high ν, the unstable solution of Eq. (21) reaches the level characteristic of the flute instability with γ=2/3, $\hat{\Gamma} \approx \sqrt{g-0.4}$.

At $g>1$ the situation is quite different (Fig. 2). First, the growth rate is positive even at ν=0; second, if g exceeds the critical value (g =1) by a factor of 2 or more, the growth rate becomes relatively independent of the artificial diffusion. This is a direct consequence of the structure of Eq. (21): the second terms varies in the limited interval, between 0 and 0.6. Thus, its effect on the growth-rate at $g>>1$ becomes small (because the first term is large).

Now we briefly consider the effect of viscous dissipation. We include it by adding a term $\eta\nabla^2\mathbf{v}$ to the right-hand side of Eq. (1a). It is easy to show that, in the limit defined by Eq. (12), it enters the problem via an additional term in the left-hand side of the dispersion relation (18) where now, instead of the term Γ^2, we have $\Gamma^2+\nu_{visc}\Gamma$, with ν_{visc} being the inverse viscous dissipation time over the scale $1/k_z$: $\nu_{visc} = \eta k_z^2/\rho_0$. Using the dimensionless expression for ν_{visc}, $\hat{\nu}_{visc} = \nu_{visc} a/2s$, one obtains, instead of Eq. (21), the following dimensionless dispersion relation:

$$\hat{\Gamma}(\hat{\Gamma} + \hat{\nu}_{visc}) = g - 1 + \frac{0.6\hat{\nu}}{\hat{\Gamma} + \hat{\nu}} \qquad (23)$$

At a zero artificial diffusivity $\hat{\nu}$=0, the viscous term does not affect the plasma stability boundary, which remains g=1. Viscosity leads just to a reduction of the growth rate, and modes mimicking "negative energy modes" are absent. However, when the artificial diffusivity is turned on, stability threshold becomes lower, and the modes with a growth rate proportional to $\hat{\nu}$ appear.

As an illustration of the effect of viscosity on the growth rate, dashed lines on Fig. 2 show the growth rate for the case where $\nu_{visc}=\nu$ (in other words, for quite high kinematic viscosity η/ρ_0, equal to the artificial diffusivity *D*). One sees that the growth rates become somewhat smaller, but no dramatic changes occur compared to the zero viscosity case.

Based on these results, we come to the following conclusions regarding the effect of the artificial diffusivity on the pressure-driven modes. The effect is unimportant if the following condition holds:

$$\Gamma >> D/\Delta^2 \tag{24}$$

where Δ is a characteristic scale-length of perturbations, and we have returned to dimensional units.

If this condition is violated, two outcomes are possible. If the contribution of the gas compressibility (the terms proportional to γ) to the growth rate is small, the growth rate remains insensitive to the artificial diffusivity (despite the fact that *D* is "large"). Conversely, if the condition (24) is violated and, at the same time, the dependence of the growth-rate on γ is substantial, the effect of the diffusivity on the instability becomes significant, leading to the broadening of the instability range and introducing modes behaving like negative-energy-modes. Adding viscous terms does not affect these conclusions, at least in a qualitative way.

Our results may serve as guidance in assessing the effect of a finite *D* on pressure-driven modes in numerical simulations. They may also provide some insights into the effect of a large *D* on other modes, e.g., current-driven modes. Here it is reasonable to assume that the role of a large *D* will be insignificant if the modes do not depend substantially on the plasma compressibility. In particular, the modes that can be described in the paraxial approximation [4] will probably be insensitive to the value of D. This suggestion, however, will have to be substantiated by a more detailed analysis.

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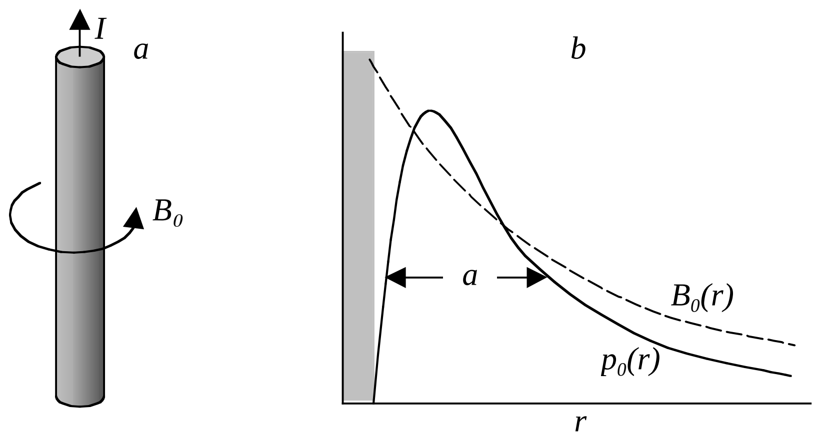

Fig. 1 A low-pressure plasma occupies an area outside a current-carrying rod (panel a). The effect of this plasma on the magnetic field is negligible, so that B_0~1/r (panel b, dashed line). The unperturbed plasma density is uniform, whereas the plasma temperature varies in the radial direction. This creates the pressure variation which is a potential source of the flute instability in the zone where $p_0' < 0$.

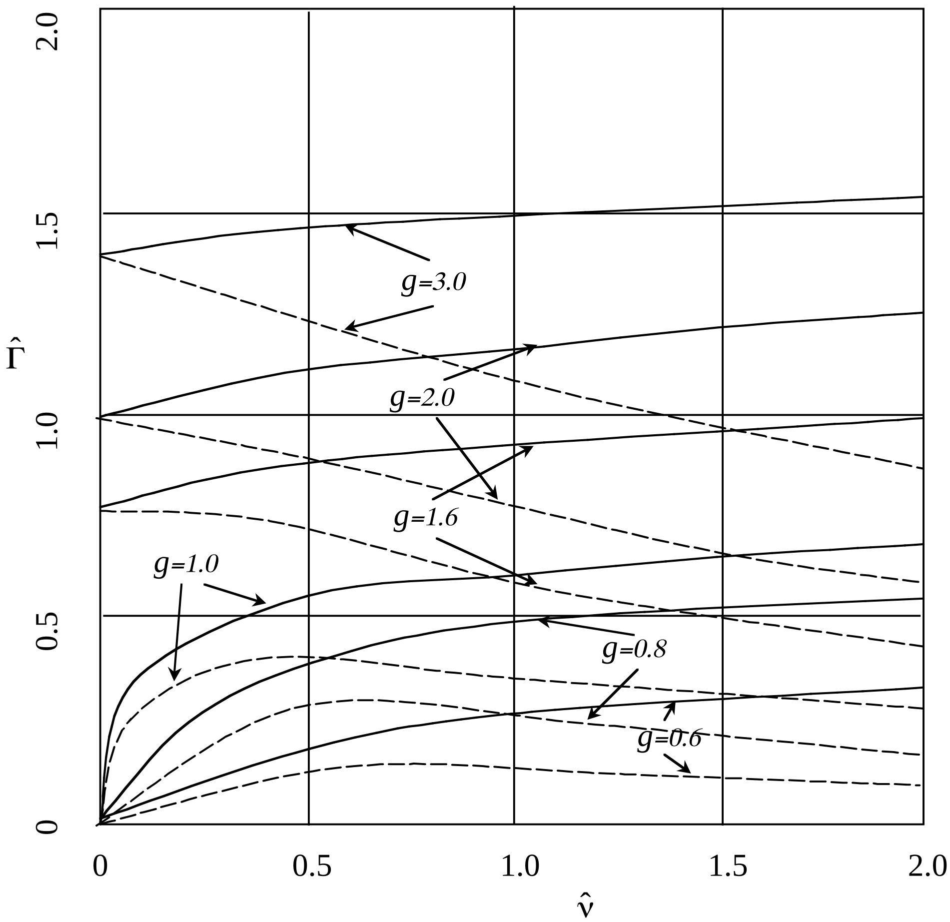

Fig. 2. Dimensionless growth rates for various values of the dimensionless gradient g. The solid curves correspond to the zero viscosity. The dashed curves correspond to the case where kinematic viscosity η/ρ_0 is equal to the artificial diffusion coefficient D.